On the decay of the smallest singular value of submatrices of rectangular matrices

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In this paper, we study the decay of the smallest singular value of submatrices that consist of bounded column vectors. We find that the smallest singular value of submatrices is related to the minimal distance of points to the lines connecting other two points in a bounded point set. Using a technique from integral geometry and from the perspective of combinatorial geometry, we show the decay rate of the minimal distance for the sets of points if the number of the points that are on the boundary of the convex hull of any subset is not too large, relative to the cardinality of the set. In the numerical or computational aspect, we conduct some numerical experiments for many sets of points and analyze the smallest distance for some extremal configurations.

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1. Introduction

In recent decades, measurements, frames, and dictionaries (see for instance, [2, 5, 24]), all of which are essentially matrices, have been studied and used in signal processing, such as compressed sensing, matrix recovery, phase retrieval, and other fields. As the main characteristics of a matrix or linear transformation, the singular values and their generalized forms have been studied in, for instance, [10, 18, 20, 23, 28]. It is not hard to see that the singular values of a matrix are determined by both the magnitudes and the angles of the row vectors of the matrix.

Rectangular matrices are of the main interest in some recent research (see, for instance, [28, 29]). Here we call a rectangular matrix a slim matrix if there are more rows than columns in the matrix. Considering the columns of a slim matrix as points in a bounded region in a plane, we show that the matrix problem can be reduced...
down to a combinatorial problem. If the magnitudes of all the rows of a rectangular matrix are bounded, we can estimate the smallest singular values of submatrices, in terms of the size of the matrix, because there are configurations of matrices whose minimal smallest singular values by the order of a power of the size with some negative exponent. Some estimates on the distances among points in a set or the distances from points to lines that connect other two points in a set of points in a bounded region are established in this paper, and the decay rate of these distances, in some sense, essentially determines the decay rate of the smallest singular values of submatrices with bounded column vectors. The combinatorial geometry problem is related to Heilbronn’s triangle problem (see, for instance, [4, 16]). There have been some work on developing algorithms to find counter example for Heilbronn’s original conjecture, but there does not appear to be any experimentable algorithm for one to find any explicit or concrete sets of points, and it would be interesting to see the optimal arrangements of \( n \) points in a square or unit disk for Heilbronn’s triangle problem and this problem respectively. However, we formulate a conjecture for a slower decay rate, which, as far as we know, is still open.

The main contribution of this paper is to show the connection between the singular value problem and a combinatorial geometry problem. Using a technique from integral geometry and from the perspective of combinatorial geometry, we show the decay rate of the minimal distance for the sets of points if the number of the points that are not on the boundary of the convex hull of any subset is not too large, relative to the cardinality of the set. We also obtain some other results regarding this combinatorial geometry problem in some cases, and so for the minimal smallest singular value of submatrices of rectangular matrices.

This paper is structured as follows: in Sec. 2, we prove some lemmas on the minimal smallest singular value of slim matrices, and particularly, we show the optimal decay rate for the base case; in Sec. 3, we prove a duality lemma for a the minimal smallest singular value of matrices of size \( n + k \) by \( n \); in Sec. 4, we undertake extensive study on the minimal smallest singular value of matrices of size \( n + 3 \) by \( n \), and we obtain some results by using a technique from integral geometry and from the perspective of combinatorial geometry; and in Sec. 5, we present some numerical experimental results.

2. Some Lemmas on the Minimal Smallest Singular Value

First, we have the following lemma.

**Lemma 2.1.** For any real matrix \( A \) of size \( N \) by \( n \) with \( N \geq n \), one has

\[
\sigma_n(A) \geq \min_{S \subseteq \{1, \ldots, N\}, |S| = n} \sigma_n(AS) \tag{2.1}
\]

and

\[
\sigma_1(A) \geq \min_{S \subseteq \{1, \ldots, N\}, |S| = n} \sigma_1(AS), \tag{2.2}
\]
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where \( \sigma_1(A) \) and \( \sigma_n(A) \) are the greatest and the least singular values of \( A \) and \( \sigma_1(A_S) \) and \( \sigma_n(A_S) \) are accordingly the greatest and the least singular values of \( A_S \) in which \( A_S \) is the submatrix of \( A \) formed by selecting the rows of \( A \) at the row numbers in \( S \).

**Proof.** For any \( S \subseteq \{1, \ldots, n+1\} \) with \( |S| = n \),
\[
\sigma_n(A_S) = \inf_{v \in \mathbb{R}^n, \|v\| = 1} \|A_Sv\|; \quad (2.3)
\]
and on the other hand,
\[
\sigma_n(A) = \inf_{V \subseteq \mathbb{R}^n, \text{dim}(V) = 1} \|A|_V\| = \inf_{v \in \mathbb{R}^n, \|v\| = 1} \|Av\|. \quad (2.4)
\]
Since \( Av \) is basically an vector extension of \( A_Sv \) for every \( v \in \mathbb{R}^n, \|v\| = 1 \), then
\[
\|A_Sv\| \leq \|Av\| \quad (2.5)
\]
for every \( v \in \mathbb{R}^n, \|v\| = 1 \). Thus, it follows from (2.3) and (2.4) that
\[
\sigma_n(A_S) \leq \sigma_n(A) \quad (2.6)
\]
for any \( S \subseteq \{1, \ldots, N\} \) with \( |S| = n \). Hence, we obtain (2.1), and similarly, we also obtain (2.2).

From the growth rate of the smallest singular value of random matrices established in [3], one can obtain that
\[
\sigma_n(A) \to (2 - \sqrt{2})\sqrt{n} \quad (2.7)
\]
for \( N = 2n \). On the other hand,
\[
\sigma_n(A_S) \leq O \left( \frac{1}{\sqrt{n}} \right). \quad (2.8)
\]

**Lemma 2.2.** For any \( n+1 \) by \( n \) matrix \( A = \begin{bmatrix} a_1 & \cdots & a_{n+1} \end{bmatrix} \) with \( \|a_i\| \leq 1, i = 1, \ldots, n+1 \), one has
\[
\min_{S \subseteq \{1, \ldots, n+1\}, |S| = n} \sigma_n(A_S) \leq \frac{1}{\sqrt{n}}. \quad (2.9)
\]

**Proof.** Since \( a_1, \ldots, a_{n+1} \) are linear dependent, there are \( c_1, \ldots, c_{n+1} \), such that
\[
\sum_{i=1}^{n+1} c_i a_i = 0 \quad (2.10)
\]
with
\[
\sum_{i=1}^{n+1} c_i^2 = 1. \quad (2.11)
\]
Without loss of generality, assume \( c_{n+1} = \min(c_1, \ldots, c_{n+1}) \). If \( c_{n+1} = 0 \), (2.9) is trivial, because there is an \( S \) such that \( A_S \) is singular. It suffices to consider the
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case of \( c_{n+1} \neq 0 \). Therefore,

\[
c_{n+1}a_{n+1} = -\sum_{i=1}^{n} c_{i}a_{i}.
\]

By (2.11),

\[
(n + 1)c_{n+1}^2 \leq \sum_{i=1}^{n+1} c_i^2 = 1.
\]

It follows that

\[
|c_{n+1}| \leq \frac{1}{\sqrt{n+1}}
\]

and furthermore

\[
\frac{||c_{n+1}a_{n+1}||}{\sqrt{1 - c_{n+1}^2}} \leq \frac{1}{\sqrt{n+1}} \frac{\sqrt{n+1}}{\sqrt{n}} = \frac{1}{\sqrt{n}}.
\]

Since

\[
\min_{S \subseteq \{1, \ldots, n+1\}, |S| = n} \sigma_n(A_S) \leq \frac{\|\sum_{i=1}^{n} c_i a_i\|}{\sqrt{\sum_{i=1}^{n} c_i^2}} = \frac{\|c_{n+1}a_{n+1}\|}{\sqrt{1 - c_{n+1}^2}},
\]

thus (2.9) follows from (2.15).

\[ \Box \]

**Remark 2.3.** However, one can have

\[
\min_{S \subseteq \{1, \ldots, n+1\}, |S| = n} \sigma_n(A_S) > \frac{1}{n}
\]

for some matrix \( A \). For example,

\[
A^T = \begin{bmatrix}
0.9969 & 0.6688 & 0.1610 \\
-0.0782 & 0.7434 & -0.9870
\end{bmatrix},
\]

we have

\[
\min_{S \subseteq \{1, \ldots, n+1\}, |S| = n} \sigma_n(A_S) = 0.6115 > \frac{1}{2}.
\]

For matrices of size \( n + 1 \) by \( n \), one can have the following.

**Lemma 2.4.** For any \( n + 2 \) by \( n \) matrix \( A = \begin{bmatrix}
a_1 \\
\vdots \\
a_{n+2}
\end{bmatrix} \) with \( \|a_i\| \leq 1, i = 1, \ldots, n + 2 \), one has

\[
\min_{S \subseteq \{1, \ldots, n+2\}, |S| = n} \sigma_n(A_S) \leq \frac{C}{n^{3/2}}
\]

for some constant \( C > 0 \).
Proof. It suffices to consider matrices of size \( n+2 \) by \( n \) with rank no less than \( n \). Then for any \( z \in \ker(A) \) with \( \|z\| = 1 \), we have

\[
\sigma_n(A_S) \leq \inf_{\mathbf{z} \in \ker(A)} \frac{\|A_S\mathbf{z}\|}{\|\mathbf{z}\|}
\]

\[
= \inf_{\mathbf{z} \in \ker(A)} \frac{\|z_1\mathbf{a}_1 + z_2\mathbf{a}_2\|}{\|\mathbf{z}\|}
\]

\[
\leq \inf_{\mathbf{z} \in \ker(A)} \frac{|z_1| \|\mathbf{a}_1\| + |z_2| \|\mathbf{a}_2\|}{\|\mathbf{z}\|}
\]

\[
\leq \inf_{\mathbf{z} \in \ker(A)} \frac{|z_1| + |z_2|}{\|\mathbf{z}\|}
\]

\[
\leq \inf_{\mathbf{z} \in \ker(A)} \frac{\sqrt{2} \sqrt{z_1^2 + z_2^2}}{\sqrt{1 - (z_1^2 + z_2^2)}}.
\]

(2.21)

where \( S = \{1, \ldots, n+2\} \setminus \{i_1, i_2\} \) for all \( 1 \leq i_1, i_2 \leq n+2 \).

Let \( \mathbf{b}_1 \) and \( \mathbf{b}_2 \) be an orthonormal basis of \( \ker(A) \), \( \mathbf{b}_1 = (b_{11}, \ldots, b_{1,n+2}) \) and \( \mathbf{b}_2 = (b_{21}, \ldots, b_{2,n+2}) \), and denote \( (\mathbf{b}_1, \mathbf{b}_2) := B \). Since \( z \in \ker(A) \) with \( \|z\| = 1 \), there exist \( t_1 \) and \( t_2 \) such that

\[
z = t_1\mathbf{b}_1 + t_2\mathbf{b}_2
\]

(2.22)

with \( t_1^2 + t_2^2 = 1 \). Therefore,

\[
\sqrt{z_1^2 + z_2^2} = \sqrt{(t_1 b_{1,i_1} + t_2 b_{2,i_1})^2 + (t_1 b_{1,i_2} + t_2 b_{2,i_2})^2}
\]

\[
= \|(t_1, t_2)B_{S^c}\|.
\]

(2.23)

Combining (2.21), we have

\[
\sigma_n(A_S) \leq C \inf_{t_1^2 + t_2^2 = 1} \|(t_1, t_2)B_{S^c}\| = C\sigma_2(B_{S^c})
\]

(2.24)

for some constant \( C > 0 \) and furthermore,

\[
\min_{S \subseteq \{1, \ldots, n+2\}, |S| = n} \sigma_n(A_S) \leq C \min_{S \subseteq \{1, \ldots, n+2\}, |S| = n} \sigma_2(B_{S^c}).
\]

(2.25)

Now let \( B = (\beta_1, \ldots, \beta_{n+2}) \) and normalize the columns of \( B \), then

\[
\sigma_2(B_{S^c}) \leq \max(\|\beta_1\|, \|\beta_2\|) \sigma_2 \left( \begin{pmatrix} \beta_{11} \\ |\beta_{11}| \end{pmatrix} \begin{pmatrix} \beta_{12} \\ |\beta_{12}| \end{pmatrix} \right).
\]

(2.26)

Now we can choose the indices \( i_1 \) and \( i_2, 1 \leq i_1, i_2 \leq n + 2 \), such that

\[
b_{1,i_1}^2 + b_{2,i_1}^2 + b_{1,i_2}^2 + b_{2,i_2}^2 = \|B_{S^c}\|_F
\]

(2.27)
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is the smallest among all pairs of indices between 1 and $n + 2$, but since

$$\sum_{i=1}^{n+2} b_{1,i}^2 + \sum_{i=1}^{n+2} b_{2,i}^2 = 2,$$

we have

$$b_{1,i_1}^2 + b_{2,i_1}^2 + b_{1,i_2}^2 + b_{2,i_2}^2 \leq \frac{4}{n+2},$$

which implies

$$\max \left( \sqrt{b_{1,i_1}^2 + b_{2,i_1}^2}, \sqrt{b_{1,i_2}^2 + b_{2,i_2}^2} \right) \leq \frac{2}{\sqrt{n+2}}.$$  \hspace{1cm} (2.30)

Therefore, by (2.26), we have

$$\min_{S \subseteq \{1, \ldots, n+2\}, |S| = n} \sigma_2(B_{S^c}) \leq \frac{2}{\sqrt{n+2}} \sigma_2 \left( \left( \frac{\beta_{i_1}}{\|\beta_{i_1}\|}, \frac{\beta_{i_2}}{\|\beta_{i_2}\|} \right) \right) \leq \frac{\sqrt{2}}{\sqrt{n+2}} \left\| \frac{\beta_{i_1}}{\|\beta_{i_1}\|} - \frac{\beta_{i_2}}{\|\beta_{i_2}\|} \right\|. \hspace{1cm} (2.31)$$

Considering the geometry of $n + 2$ vectors on the unit circle and choose the closest two vectors among the $n + 2$ unit vectors, we know

$$\left\| \frac{\beta_{i_1}}{\|\beta_{i_1}\|} - \frac{\beta_{i_2}}{\|\beta_{i_2}\|} \right\| \leq 2 \sin \frac{\pi}{n+2}.$$  \hspace{1cm} (2.32)

Next, we will show the following inequality

$$\min_{S \subseteq \{1, \ldots, n+2\}, |S| = n} \sigma_2(B_{S^c}) \leq \frac{2\sqrt{2}}{\sqrt{n+2}} \sin \frac{\pi}{n+2}. \hspace{1cm} (2.33)$$

Suppose that

$$\sigma_2(B_{S^c}) \geq \frac{2\sqrt{2}}{\sqrt{n+2}} \sin \frac{\pi}{n+2}$$

for all $S \subseteq \{1, \ldots, n+2\}$ with $|S| = n$. For any

$$B_{S^c} = (\beta_i, \beta_j),$$

we have

$$\sigma_2(B_{S^c}) \leq \left\| \frac{\beta_i}{\|\beta_i\|} - \frac{\beta_j}{\|\beta_j\|} \right\| \leq \min(\|\beta_i\|, \|\beta_j\|) \left\| \frac{\beta_i}{\|\beta_i\|} - \frac{\beta_j}{\|\beta_j\|} \right\| \hspace{1cm} (2.36)$$

for $1 \leq i < j \leq n + 2$. We can actually arrange the indices in $\beta_i, 1 \leq i \leq n + 2$, so that $\frac{\beta_i}{\|\beta_i\|}, 1 \leq i \leq n + 2$, are in the counterclockwise order in the unit disk.
By (2.28), we know that
\[ \sum_{i=1}^{n+2} \| \beta_i \|^2 = 2, \] (2.37)
and since any chord is shorter than its corresponding arc on a circle, we have that
\[ \sum_{i=1}^{n+2} \left| \frac{\beta_i}{\| \beta_i \|} - \frac{\beta_{i+1}}{\| \beta_{i+1} \|} \right| \leq 2\pi, \] (2.38)
assuming \( \beta_{n+3} = \beta_1 \). From (2.36),
\[ \min_{S \subseteq \{1, \ldots, n+2\}, |S| = n} \sigma_2(B_{S^c}) \leq \min_{1 \leq i < j \leq n+2} \left( \min(\| \beta_i \|, \| \beta_j \|) \right) \frac{\| \beta_i \|}{\| \beta_j \|} \left( \| \beta_i \| - \| \beta_j \| \right), \] (2.39)
From (2.37) and (2.38), one can obtain that
\[ \min_{1 \leq i \leq n+2} \left( \| \beta_i \| \left( \frac{\beta_i}{\| \beta_i \|} - \frac{\beta_{i+1}}{\| \beta_{i+1} \|} \right) \right) \leq \frac{1}{n+2} \sum_{i=1}^{n+2} \left( \| \beta_i \| \left( \frac{\beta_i}{\| \beta_i \|} - \frac{\beta_{i+1}}{\| \beta_{i+1} \|} \right) \right) \]
\[ \leq \frac{2\sqrt{2}\pi}{(n+2)^{3/2}} \] (2.40)
and then (2.20) follows.

3. Duality Lemma for Matrices of Size \( n + k \) by \( n \)

First we have the following duality lemma in general.

**Lemma 3.1.** For any matrix \( A \) of size \( m \) by \( n \) with \( m \geq n \) and all rows normalized to 1, one has
\[ \min_{S \subseteq \{1, \ldots, m\}, |S| = n} \sigma_n(A_S) \leq C \min_{T \subseteq \{1, \ldots, n\}, |T| = m-n} \sigma_n(B_T) \] (3.1)
for some constant \( C > 0 \), where \( B \) consists of any orthogonal basis of \( \ker(A) \).

**Proof.** Then for any \( z \in \ker(A) \) with \( \| z \| = 1 \), we have
\[ \sigma_n(A_S) \leq \inf_{z \in \ker(A)} \frac{\| A_S z_S \|}{\| z_S \|} \]
\[ = \inf_{z \in \ker(A)} \frac{\| z_1 a_1 + z_2 a_2 \|}{\| z_S \|} \]
\[ \leq \inf_{z \in \ker(A)} \frac{|z_1| \| a_1 \| + |z_2| \| a_2 \|}{\| z_S \|} \]
\[ \leq \inf_{z \in \ker(A)} \frac{\| z_{S^c} \|}{\| z_S \|} \]
\[ \leq \inf_{z \in \ker(A)} \frac{\sqrt{2} \| z_{S^c} \|}{\sqrt{1 - \| z_{S^c} \|^2}}. \] (3.2)
Let $b_1$ and $b_2$ be an orthonormal basis of $\ker(A)$, $b_1 = (b_{11}, \ldots, b_{1n+2})$ and $b_2 = (b_{21}, \ldots, b_{2n+2})$, and denote $\{b_1, b_2\} := B$. Since $z \in \ker(A)$ with $\|z\| = 1$, there exist $t_1$ and $t_2$ such that

$$z = tB_S$$

with $t_1^2 + t_2^2 = 1$. Therefore,

$$\|z_S\| = \|tB_S\|.$$  

(3.4)

Combining (3.2), we have

$$\sigma_n(A_S) \leq \inf_{t \in \mathbb{R}^{m-n}} \|tB_S\| = C\sigma_{m-n}(B_S)$$

(3.5)

for some constant $C > 0$, where $S^{m-n}$ is the unit sphere in $\mathbb{R}^{m-n+1}$, and furthermore,

$$\min_{S \subseteq \{1, \ldots, m\}, |S|=n} \sigma_n(A_S) \leq C \min_{T \subseteq \{1, \ldots, m\}, |T|=m-n} \sigma_n(B_T).$$

(3.6)

Remark 3.2. In matrix theory and operator theory, the image of an operator is regarded as to be dual its kernel or null space. Here this duality is in a similar essence to relationship between the restricted isometry property, Johnson–Lindenstrauss embedding, and the null space property in signal processing, including compressed sensing, phase retrieval, and others (see for instance, [17, 26, 27]).

4. Decay Rate for Matrices of Size $n + 3$ by $n$

Let $P_1, \ldots, P_n$ be in the unit disk on the plane, and $d(i, j, k)$ be the distance of the point $P_i$ to the line connecting other two points $P_j$ and $P_k$, $1 \leq i, j, k \leq n$. In this section, we want to study the decay of $\min_{1 \leq i, j, k \leq n} d(i, j, k)$, as $n \to \infty$.

First, let us prove the following lemma on the decay order of at least $O(\frac{1}{n})$.

Lemma 4.1. Let $P_1, \ldots, P_n$ be a set of points in the unit disk on the plane. Suppose that $P_1, \ldots, P_n$ are on the boundary of the convex hull of the point set $\{P_1, \ldots, P_n\}$ and $d(i, j, k)$ is the distance of the point $P_i$ to the line connecting other two points $P_j$ and $P_k$, $1 \leq i, j, k \leq n$, then

$$\min_{1 \leq i, j, k \leq n} d(i, j, k) \leq C \frac{1}{n}$$

for some absolute constant $C, C > 0$, independent of $n$.

Proof. Let us cover the unit disk by parallel stripes of width $\frac{2}{n}$, then the unit disk can be covered by $\left[\frac{n}{2}\right]$ such stripes. By the pigeonhole principle, there exist at least three points $P_{i_0}, P_{j_0}$ and $P_{k_0}$ which locate in the same strip, thus we have

$$\min_{1 \leq i, j, k \leq n} d(i, j, k) \leq d(i_0, j_0, k_0) \leq \frac{8}{n}.$$  

(4.2)
Next, we prove the following lemma.

**Lemma 4.2.** Let $P_1, \ldots, P_n$ be a set of points in the unit disk on the plane. Suppose that $P_1, \ldots, P_n$ are on the boundary of the convex hull of the point set $\{P_1, \ldots, P_n\}$ and $d(i, j, k)$ is the distance of the point $P_i$ to the line connecting other two points $P_j$ and $P_k$, $1 \leq i, j, k \leq n$, then

$$\min_{1 \leq i,j,k \leq n} d(i, j, k) \leq \frac{C}{n^2}$$

(4.3)

for some absolute constant $C, C > 0$, independent of $n$.

**Proof.** Without loss of generality, we assume that the points $P_1, P_2, \ldots, P_n$ are in the counterclockwise order in the unit disk. First, if $P_1, \ldots, P_n$ are the vertices of a convex polygon $\mathbf{P}$, then by the Crofton formula in integral geometry or geometric probability (see for instance [15, 22, 30]),

$$\text{perimeter}(\mathbf{P}) = \frac{1}{2} \int_0^{2\pi} \int_0^1 n_P(\theta, r) dr d\theta,$$

(4.4)

where $n_P(\theta, r)$ is the intersection number of the polygon and the oriented line which has a distance $r$ to the origin and has an angle $\theta$ to the positive horizontal axis. Let $\mathbf{C}$ be the unit circle, again by the Crofton formula, we know

$$\text{perimeter}(\mathbf{C}) = \frac{1}{2} \int_0^{2\pi} \int_0^1 n_C(\theta, r) dr d\theta.$$

(4.5)

But since the polygon $\mathbf{P}$ is convex, then

$$n_P(\theta, r) \leq 2 = n_C(\theta, r),$$

(4.6)

and it follows from (4.4) and (4.5) that

$$\text{perimeter}(\mathbf{P}) \leq \text{perimeter}(\mathbf{C}) = \frac{1}{2} \int_0^{2\pi} \int_0^1 2dr d\theta = 2\pi.$$

(4.7)

Thus the sum of the boundary edges of the polygon

$$\sum_{i=1}^n \|P_i P_{i+1}\| \leq 2\pi.$$

(4.8)

Now let us connect the vertices by edges $P_1 P_3, P_2 P_4, \ldots, P_{n-1} P_1$, and $P_n P_2$, which form angles that are denoted by $\angle$, then we have

$$\sum_{i=1}^n (\angle P_{i+2} P_i P_{i+1} + \angle P_{i+1} P_{i+3} P_i) = n\pi - (n-2)\pi = 2\pi$$

(4.9)

assuming $P_{n+1} = P_1$ and $P_{n+2} = P_2$, because there are $n$ triangles and the sum of the interior angles of the polygon is $(n-2)\pi$. Furthermore, since
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\[ \sum_{i=1}^{n} (\sin(\angle P_{i+2}P_iP_{i+1}) + \sin(\angle P_{i+1}P_{i+2}P_i)) \]

\[ \leq \sum_{i=1}^{n} (\angle P_{i+2}P_iP_{i+1} + \angle P_{i+1}P_{i+2}P_i) \]  

(4.10)

therefore, we have

\[ \sum_{i=1}^{n} (\sin(\angle P_{i+2}P_iP_{i+1}) + \sin(\angle P_{i+1}P_{i+2}P_i)) \leq 2\pi. \]  

(4.11)

By Cauchy–Schwarz inequality,

\[ \sum_{i=1}^{n} \|P_iP_{i+1}\|^\frac{1}{2} (\sin\frac{1}{2}(\angle P_{i+2}P_iP_{i+1}) + \sin\frac{1}{2}(\angle P_{i+1}P_{i+2}P_i)) \]

\[ \leq \left( \sum_{i=1}^{n} 2 \|P_iP_{i+1}\|^\frac{1}{2} \sum_{i=1}^{n} (\sin(\angle P_{i+2}P_iP_{i+1}) + \sin(\angle P_{i+1}P_{i+2}P_i)) \right). \]  

(4.12)

It follows from (4.8) and (4.11) that

\[ \sum_{i=1}^{n} \|P_iP_{i+1}\|^\frac{1}{2} (\sin\frac{1}{2}(\angle P_{i+2}P_iP_{i+1}) \]

\[ + \sin\frac{1}{2}(\angle P_{i+1}P_{i+2}P_i)) \leq 4\pi \cdot 2\pi = 8\pi^2. \]  

(4.13)

Since there are actually 2n terms in the above sum, then we have

\[ \min_{1 \leq i \leq n} \|P_iP_{i+1}\|^\frac{1}{2} \sin\frac{1}{2}(\angle P_{i+2}P_iP_{i+1}) \leq \frac{8\pi^2}{2n} = \frac{4\pi^2}{n}. \]  

(4.14)

or

\[ \min_{1 \leq i \leq n} \|P_iP_{i+1}\|^\frac{1}{2} \sin\frac{1}{2}(\angle P_{i+1}P_{i+2}P_i) \leq \frac{8\pi^2}{2n} = \frac{4\pi^2}{n}. \]  

(4.15)

Notice that

\[ d(i+1, i, i+2) = \|P_iP_{i+1}\| \sin(\angle P_{i+2}P_iP_{i+1}) \]

\[ = \|P_{i+1}P_{i+2}\| \sin(\angle P_{i+1}P_{i+2}P_i), \]  

(4.16)

thus by (4.14) and (4.15) we know that

\[ \min_{1 \leq i \leq n} d(i+1, i, i+2) \leq \frac{16\pi^4}{n^2}. \]  

(4.17)

and the claim in 4.2 follows, in the case that \(P_1, \ldots, P_n\) are the vertices of a convex polygon.

Second, if a point is on the boundary edges of a convex hull of the point set but is not a vertices of the convex polygon, then the distance of the point to the
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Remark 4.3. In the proof of the above lemma, we have used a technique from integral geometry. For generalized theory of it, one can refer to, for instance, [1, 11, 21, 31].

From this lemma, we can derive the following corollary immediately.

Lemma 4.4. Let \( P_1, \ldots, P_n \) be a set of points in the unit disk on the plane. Suppose that \( P_1, \ldots, P_n, 0 \leq s \leq n - 4 \), are on the boundary of the convex hull of the point set \( \{P_1, \ldots, P_n\} \) and \( d(i, j, k) \) is the distance of the point \( P_i \) to the line connecting other two points \( P_j \) and \( P_k \), \( 1 \leq i, j, k \leq n \), then

\[
\min_{1 \leq i,j,k \leq n} d(i, j, k) \leq \frac{C}{(n-s)^2} \tag{4.18}
\]

for some absolute constant \( C, C > 0 \), independent of \( n \). In particular, if \( s \leq \lfloor \frac{n}{2} \rfloor \), we have

\[
\min_{1 \leq i,j,k \leq n} d(i, j, k) \leq \frac{4C}{n^2} \tag{4.19}
\]

More generally, if \( s \leq \lfloor \lambda n \rfloor \) for some absolute constant \( \lambda, \lambda > 0 \), independent of \( n \), then

\[
\min_{1 \leq i,j,k \leq n} d(i, j, k) \leq \frac{C}{n^2} \tag{4.20}
\]

for some absolute constant \( C, C > 0 \), independent of \( n \).

Remark 4.5. Note that \( s \leq n - 4 \), because by the Sylvester–Gallai theorem (see for instance [6, 14]), if all the points are not collinear, there is a line which passes through exactly two of the points, but (4.3) will trivially hold if there exist three points in the point set that are collinear and here we only need to consider the sets of \( n \) points which have exactly \( \frac{n(n-1)}{2} \) ordinary lines, on which one can refer to [9], and also by the Erdős–Szekeres theorem (see for instance [7, 25]), any set of \( n \) generic points, \( n \geq 4 \), in the plane has at least four points that are the vertices of a convex quadrilateral.

In [8, 13], a set of \( 2^{n-2} \) points that contains no convex \( n \)-gon was constructed. We will analyze the minimal distance \( \min_{1 \leq i,j,k \leq n} d(i, j, k) \) for this extremal case. Let

\[
S_{k,l} := \left\{(x, y_{k,l}(x)) : 1 \leq x \leq \binom{k + l - 2}{k - 1}\right\} \tag{4.21}
\]

and define \( y_{k,l}(x) \) inductively as follows:

(i) \( y_{k,1}(1) = y_{1,1}(1) = 1 \);
(ii) if \( k > 1, l > 1 \), then

\[
y_{k,l}(x) = y_{k,l-1}(x) \tag{4.22}
\]
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for $1 \leq x \leq \binom{k+l-3}{k-1}$ and

$$y_{k,l}(x) = y_{k-1,l} \left( x - \binom{k+l-3}{k-1} \right) + \alpha_{k,l} \tag{4.23}$$

for $\binom{k+l-3}{k-1} < x \leq \binom{k+l-2}{k-1}$, where

$$\alpha_{k,l} = \binom{k+l-2}{k-1} \max \left( y_{k-1,l} \left( \binom{k+l-3}{k-1} \right), y_{k-1,l} \left( \binom{k+l-3}{k-2} \right) \right). \tag{4.24}$$

From the inductive definition of $y_{k,l}(x)$, we know that $y_{k,l}$ linearly depends on $y_{k-1,l}$ and $y_{k-1,l-1}$. By [13], $y_{k,l}(x)$ is monotone increasing with respect to $x$ for $1 \leq x \leq \binom{k+l-3}{k-1}$. But $y_{k,l}(x)$ increases dramatically when $x$ becomes large.

Now let us consider $S_{n,n}$, the cardinality of $S_{n,n}$

$$|S_{n,n}| = \binom{2n-2}{n-1}. \tag{4.25}$$

To preserve the convexity and concavity of subsets in $S_{n,n}$ and confine it into the unit square, we use a similarity transformation

$$T = \begin{pmatrix} \frac{(n-1)!^2}{(2n-2)!} & 0 \\ 0 & 1/y_{n,n}(\binom{2n-2}{n-1}) \end{pmatrix}, \tag{4.26}$$

and then $T(S_{n,n}) \subset [0,1]^2$. Since $S_{n,n}$ is one of the components of the set of $N = 2^{2n-2}$ points $R_N$ that contains no convex $n$-gon, $T(S_{n,n})$ is the one of the components of the set of $N = 2^{2n-2}$ points in $[0,1]^2$ that contains no convex $n$-gon. From Fig. 1, we can see that the minimal distance $\min_{1 \leq i,j,k \leq n} d(i,j,k)$ in $S_{n,n}$ multiplied by $N^2 = 2^{4n-4}$ is very likely bounded, that implies the minimal distance $\min_{1 \leq i,j,k \leq n} d(i,j,k)$ in the set of $N = 2^{2n-2}$ points $R_N$ should decay at the rate of at least $O(\frac{1}{N^2})$.

Considering the configurations of $n$ points in the unit disk, we have the following lemma first.

**Lemma 4.6.** Let $D$ be the unit disk, then

$$\min_{1 \leq i,j,k \leq n} d(v_i, v_j, v_k) \leq 2 \sin^2 \frac{\pi}{n} \tag{4.27}$$

for all $v_1, v_2, \ldots, v_n \in D$ for $n = 3$ and 4. Therefore

$$\min_{1 \leq i,j,k \leq n} d(v_i, v_j, v_k) \leq \frac{2\pi^2}{n^2} \tag{4.28}$$

for all $v_1, v_2, \ldots, v_n \in D$ for $n = 3$ and 4.
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Fig. 1. Plotted above are the smallest distances in $S_{n,n}$ multiplied by $2^{4n-4}$.

**Proof.** For $n = 3$, there are three points $v_1, v_2$ and $v_3$ in $\mathbb{D}$. Without loss of generality, we can assume that the side $\overrightarrow{v_1v_2}$ is the longest side and $v_1$ and $v_2$ lie on the boundary of $\mathbb{D}$, denoted as $\partial \mathbb{D}$, because one can use translations and rotations.

Let $v_3'$ be the intersection of the line parallel to the side $\overrightarrow{v_1v_2}$ and its perpendicular bisector. Then we have

$$d(v_3', v_1, v_2) = d(v_3, v_1, v_2)$$

and then the minimal heights of the triangle $\triangle v_1v_2v_3$ and $\triangle v_1v_2v_3'$ are equal, because

$$||v_1v_3'|| = ||v_2v_3'|| \leq \max(||v_2v_3'||, ||v_1v_3'||) \leq ||v_1v_2||,$$

in other words, $\overrightarrow{v_1v_2}$ is also the longest side of $\triangle v_1v_2v_3'$, and the areas of the triangle $\triangle v_1v_2v_3$ and $\triangle v_1v_2v_3'$ are equal.

Now, let us move $v_3'$ along the perpendicular bisector of the side $\overrightarrow{v_1v_2}$ toward the direction in which the height increases, until it touches the boundary $\partial \mathbb{D}$ at a point denoted by $v_3''$. Let the distance from a point $v_3(t)$ on the perpendicular bisector of the side $\overrightarrow{v_1v_2}$ to the side $\overrightarrow{v_1v_2}$ be $t$, then the minimal height of the triangle $\triangle v_1v_2v_3(t)$

$$\max \left( \frac{t||v_1v_2||}{||v_1v_3'||}, \sqrt{t^2 + \frac{||v_1v_2||^2}{4}} \right) = \begin{cases} t, & 0 < t \leq \frac{\sqrt{3}||v_1v_2||}{2} \\ \frac{t||v_1v_2||}{\sqrt{t^2 + \frac{||v_1v_2||^2}{4}}}, & t > \frac{\sqrt{3}||v_1v_2||}{2} \end{cases} (4.31)$$

increases as $t$ increases. Thus the minimal height of the triangle $\triangle v_1v_2v_3''$ is greater than or equal to that of the triangle $\triangle v_1v_2v_3'$.

Then, we can do a regularization for the $\triangle v_1v_2v_3''$ whose vertices all lie on $\partial \mathbb{D}$. If one of the vertices does not bisect the arc ending with the other two vertices, and
without loss of generality, we can assume that \( v_4' \) does not bisect the arc ending with \( v_1 \) and \( v_2 \), then move \( v_4' \) to the midpoint of the arc, and then the new triangle lying on \( \partial \mathbb{D} \) has a great minimal height, by comparing trigonometric functions. Thus, the equilateral triangle lying on \( \partial \mathbb{D} \) has the greatest minimal height. This finishes the proof for the case of \( n = 3 \).

For \( n = 4 \), there are two cases to consider, but we will be able to find the maximum of the minimal heights for both cases. The first case is that one of the four points is in the interior of the convex hall of the other three points. Let us assume that, \( v_4 \) is in the interior of the convex hall of the other three points \( v_1, v_2, v_3 \). Then if we fix \( v_1, v_2, v_3 \), the maximum of the minimal heights for this case is reached when \( v_4 \) is at the center of the incircle of the triangle \( \triangle v_1 v_2 v_3 \), because otherwise, the minimal height \( \min_{1 \leq i, j, k \leq 4} d(v_i, v_j, v_k) \) would be less than the radius of the incircle of the triangle \( \triangle v_1 v_2 v_3 \). Using an argument similar to the case of \( n = 3 \), we can show that in this case,

\[
\min_{1 \leq i, j, k \leq 4} d(v_i, v_j, v_k) \leq \frac{1}{2} \leq 2 \sin^2 \frac{\pi}{4}.
\]

The second case is that the four points are all on the boundary of the convex hull of the point set \( \{v_1, v_2, v_3, v_4\} \). One can always find a rectangle \( R \) inside the quadrilateral which has the same minimal height of the triangles of the rectangle \( R \) as the minimal height of the triangles of the quadrilateral. By translations and dilations, one can obtain another rectangle \( R' \) on \( \partial \mathbb{D} \) of which the minimal height of the triangles is not less than minimal height of the triangles of the rectangle \( R \). Through maximizing a simple function, one can get that

\[
\min_{1 \leq i, j, k \leq 4} d(v_i, v_j, v_k) \leq 2 \sin^2 \frac{\pi}{4}.
\]

in this case.

In general, if all the points are on the boundary of the convex hull of the point set \( \{v_1, v_2, \ldots, v_n\} \), we have the following.

**Lemma 4.7.** Let \( \mathbb{D} \) be the unit disk, then

\[
\min_{1 \leq i, j, k \leq n} d(v_i, v_j, v_k) \leq 2 \sin^2 \frac{\pi}{n}.
\]

for all \( v_1, v_2, \ldots, v_n \in \mathbb{D} \) if all the points are on the boundary of the convex hull of the point set \( \{v_1, v_2, \ldots, v_n\} \). Therefore

\[
\min_{1 \leq i, j, k \leq n} d(v_i, v_j, v_k) \leq \frac{2\pi^2}{n^2}
\]

for all \( v_1, v_2, \ldots, v_n \in \mathbb{D} \) if all the points are on the boundary of the convex hull of the point set \( \{v_1, v_2, \ldots, v_n\} \).

**Proof.** If all the points are on the boundary of the convex hull of the point set \( \{v_1, v_2, \ldots, v_n\} \), we can move the points \( \{v_1, v_2, \ldots, v_n\} \) toward the boundary and have a convex \( n \)-gon whose vertices \( \{v'_1, v'_2, \ldots, v'_n\} \) are on \( \partial \mathbb{D} \) whose perimeter is
no less than that of the \( n \)-gon \( \{v_1, v_2, \ldots, v_n\} \), because suppose that a vertex \( v_i \) is not on the boundary \( \partial D \), then the level set

\[
\{ v \in D : \|v v_{i0}^{-1}\| + \|v v_{i0+1}\| = \|v v_{i0} v_{i0-1}\| + \|v v_{i0} v_{i0+1}\| \},
\]

(4.36)

where \( v_{i0}^{-1} \) and \( v_{i0+1} \) (assuming \( v_{n+1} = v_1 \)) are the adjacent vertices of \( v_{i0} \), is an ellipse. Connect the center of the disk \( D \) and \( v_{i0} \) by a ray and extend the ray till it intersects the boundary \( \partial D \) at \( v_{i0}' \), then

\[
\|v v_{i0} v_{i0-1}\| + \|v v_{i0} v_{i0+1}\| \geq \|v v_{i0} v_{i0-1}\| + \|v v_{i0} v_{i0+1}\|.
\]

(4.37)

Thus

\[
\text{perimeter}(v_1' v_2' \ldots v_n') \geq \text{perimeter}(v_1 v_2 \ldots v_n).
\]

(4.38)

Let \( \theta_i \) be the central angle of the chord \( v_i' v_{i+1}' \), assuming \( v_{n+1}' = v_1' \). Then

\[
\text{perimeter}(v_1' v_2' \ldots v_n') = 2 \sum_{i=1}^{n} \sin \frac{\theta_i}{2} \leq 2n \sin \left( \frac{\sum_{i=1}^{n} \theta_i}{2n} \right) = 2n \sin \frac{\pi}{n}
\]

(4.39)

by the concavity of the sine function. Combining (4.38) and (4.39), we have

\[
\text{perimeter}(v_1' v_2' \ldots v_n') \leq 2n \sin \frac{\pi}{n}
\]

(4.40)

Let us denote the angle between \( v_{i1} v_{i+1} \) and \( v_{i1} v_{i+2} \) by \( \alpha_i \) and the angle between \( v_{i+2} v_{i+1} \) and \( v_{i+2} v_{i} \) by \( \beta_i \), assuming \( v_{n+1} = v_1 \) and \( v_{n+2} = v_2 \), then

\[
\sum_{i=1}^{n} \alpha_i + \sum_{i=1}^{n} \beta_i = 2\pi
\]

(4.41)

and furthermore, we have

\[
\sum_{i=1}^{n} \sin \alpha_i + \sum_{i=1}^{n} \sin \beta_i \leq 2n \sin \left( \frac{\sum_{i=1}^{n} \alpha_i + \sum_{i=1}^{n} \beta_i}{2n} \right) = 2n \sin \frac{\pi}{n}
\]

(4.42)

again by the concavity of the sine function. Let \( s_i := \|v_i v_{i+1}\| \), \( x_i := \sin \alpha_i \) and \( y_i := \sin \beta_i \) for \( i = 1, \ldots, n \), then

\[
\sum_{i=1}^{n} x_i + \sum_{i=1}^{n} y_i \leq 2n \sin \frac{\pi}{n}
\]

(4.43)

and

\[
\sum_{i=1}^{n} s_i \leq 2n \sin \frac{\pi}{n}
\]

(4.44)

Define

\[
F := \sum_{i=1}^{n} s_i (x_i + y_i) - \lambda \left( \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} y_i - 1 \right) - \mu \left( \sum_{i=1}^{n} s_i - c_1 \right),
\]

(4.45)
where $0 \leq c_1 \leq 2n \sin \frac{\pi}{n}$ and $0 \leq c_2 \leq 2n \sin \frac{\pi}{n}$. Solving the system of equations,
\[
\sum_{i=1}^{n} x_i + \sum_{i=1}^{n} y_i = c_1, 
\]
and
\[
\sum_{i=1}^{n} s_i = c_2,
\]
that is $\lambda = s_i$, and
\[
\partial x_i F = 0,
\]
that is
\[
\partial s_i F = 0,
\]
we get $s_i = \frac{c_2}{n}$ and
\[
x_i + y_i = \frac{c_1}{n}
\]
for $i = 1, \ldots, n$. By the method of Lagrange multipliers with multiple constraints (see for instance, [12, 19]),
\[
\sum_{i=1}^{n} s_i(x_i + y_i) \leq \frac{c_1 c_2}{n} \leq 4n \sin^2 \frac{\pi}{n},
\]
which implies
\[
2n \min \left( \min_{1 \leq i \leq n} s_i x_i, \min_{1 \leq i \leq n} s_i y_i \right) \leq 4n \sin^2 \frac{\pi}{n}.
\]
Thus, there exists an $i_0$, $1 \leq i_0 \leq n$, such that either
\[
s_{i_0} x_{i_0} \leq 2 \sin^2 \frac{\pi}{n}
\]
or
\[
s_{i_0} y_{i_0} \leq 2 \sin^2 \frac{\pi}{n}
\]
in other words, either
\[
\|v_{i_0} v_{i_0+1}\| \sin \alpha_{i_0} \leq 2 \sin^2 \frac{\pi}{n}
\]
or
\[
\|v_{i_0} v_{i_0+1}\| \sin \beta_{i_0} \leq 2 \sin^2 \frac{\pi}{n}
\]
which implies (4.27) as desired. \hfill \Box
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Now, let us consider the complementary probability that any point does not fall into the stripes around the lines connecting the preceding points. To obtain the conditional probability each time when a point is dropped into the disk, one needs to have a lower bound of the covering area. This approach calculates the covering area of the stripes which have overlaps, but to find the covering area, it would depend on the configurations. For example, when the fourth point is dropped into the disk, there would be a difference on the next conditional probability whether the point is dropped into the interior of the region formed by the three preceding points or the exterior of the region. More precisely, if there are four random points, then there will be seven overlaps (including one of them overlapped by three stripes) among the stripes if three points form a triangle whose interior contains the other point, whereas there will be only four overlaps among the stripes if four points form a quadrilateral. So the covering area depends on the configuration of the points in the unit square or unit disk.

Furthermore, one would need to have a significantly small probability estimate on the minimal distance greater than \( \frac{C}{n} \) or more strongly \( \frac{C}{\sqrt{n}} \) in order to show that the probability that the minimal distance is less than \( \frac{C}{n} \) is significantly high. Thus, if one uses the probability approach, the covering area of the stripes may be estimated. But the obstruction caused by configurations or convexity is still the main hard part to solve the problem completely by soft analysis or by quasi-exact hard analysis.

Let us look into the subdivisions of the unit square now. Let \( S \) be a set of \( n \) points in the unit square. Let \( q_n \) be the maximum of the minimal distance from any point of \( S \) to the line joining any other two points of \( S \), in which the maximum is taken over all configurations of \( n \) points in the unit square, and \( p_n = nq_n \). Suppose \( S_0 \) is the configuration that achieves the maximum, and divide the unit square into \( 4^k \) sub-regions of equal area and equal shape, by using the midpoints of the edges, with a suitable arrangement of the boundaries so that every point belongs to only one sub-square. We have the following lemma regarding the behavior of \( p_n \).

**Lemma 4.8.** Suppose that a sub-region contains not more than \( \frac{n}{4^{k+1}} \) points of \( S_0 \) for some \( l > 0 \). Then there exists an \( n_1 \), such that \( \frac{(4^k + l - 1)n}{2(4^k + l - 1)n_1} < n_1 < n \) and \( p_n \leq \frac{(4^k - 1)(4^k + l)}{2(4^k + l - 1)} p_{n_1} \).

**Proof.** By pigeonhole principle, there exists an sub-region \( Q \) that contains at least \( \frac{(4^k + l - 1)n}{2(4^k + l - 1)} \) points of \( S_0 \). Let \( n_1 \) be the number of points of \( S_0 \) that falls into \( Q \). Then

\[
q_n \leq \min_{v_i, v_j, v_k \in Q} d(v_i, v_j, v_k) \leq \frac{1}{2^k}q_{n_1} = \frac{1}{2^k}p_{n_1} \leq \frac{(4^k - 1)(4^k + l)}{2^k(4^k + l - 1)n} p_{n_1}.
\]

Thus it follows that

\[
p_n \leq \frac{(4^k - 1)(4^k + l)}{2^k(4^k + l - 1)} p_{n_1}.
\]
Let us continue considering the subdivisions of the unit square.

**Lemma 4.9.** Let \( v_1, \ldots, v_n \) be a set of \( n \) points in the unit square on the plane, and connect all pairs of points by line segments. Given any \( \varepsilon, 0 < \varepsilon < 1 \), there exist more than \( \left\lfloor \frac{n^2}{4} - \frac{1}{\varepsilon^2} \right\rfloor \) distinct line segments whose length is less than \( \varepsilon \).

**Proof.** Let us divide the unit square into \( 4k \) sub-squares of equal area, by using the midpoints of the edges, with a suitable arrangement of the boundaries so that every point belongs to only one sub-square and connect every pair of points in the same sub-square by line segment.

For any given \( \varepsilon, 0 < \varepsilon < 1 \), there exists an \( k \) such that
\[
\frac{\sqrt{2}}{\varepsilon} < 2^k < \frac{2\sqrt{2}}{\varepsilon}.
\] (4.60)

Let \( n_i \) be the number of points in the \( i \)th sub-square, \( i = 1, \ldots, 4k \), then \( n = \sum_{i=1}^{4k} n_i \), and the total number of the line segments in the sub-squares is
\[
\sum_{i=1}^{4k} \frac{n_i(n_i - 1)}{2} \geq \frac{1}{2} \sum_{i}^{} (n_i - 1) = \frac{n - 4^k}{2},
\] (4.61)

since \( \frac{n_i(n_i - 1)}{2} = 0 \) if \( n_i = 0 \) or 1. Furthermore, by (4.60),
\[
\frac{n - 4^k}{2} > \frac{n}{2} - \frac{4}{\varepsilon^2}.
\] (4.62)

Thus, the total number of line segments in the sub-squares is greater than \( \left\lfloor \frac{n^2}{4} - \frac{1}{\varepsilon^2} \right\rfloor \), and the length of each line segment is less than \( \varepsilon \), since the length of each side of the sub-squares is \( \frac{\sqrt{2}}{2^k} \) that is less than \( \varepsilon \) by (4.60).

On the angles, one has the following lemma.

**Lemma 4.10.** For any \( \alpha > 0 \), among the angles between the \( \left\lfloor \frac{n^2}{4} - \frac{1}{\varepsilon^2} \right\rfloor \) distinct lines, there exist at least \( \left\lfloor \alpha \left( \frac{n^2}{4} - \frac{1}{\varepsilon^2} \right) \right\rfloor \) angles less than \( \alpha \).

**Proof.** Take any point in the plane as the vertex of the angle \( \pi \) and divide the angle into \( \left\lfloor \frac{n}{\alpha} + 1 \right\rfloor \) smaller angles of equal degree. We can do parallel transports on the lines so that they pass through the vertex of the angle \( \pi \). Then by the pigeonhole principle, there must be \( \left\lfloor \frac{\alpha(\alpha^2 - 8)}{2(\alpha + \pi)} \right\rfloor \) lines falling into the same angle, which is less than \( \alpha \).

Considering the edge and angle, one has the following.

**Lemma 4.11.** If the smallest angle and edge are adjacent, then
\[
\min_{1 \leq i < j \leq n} d(v_i, v_j, v_k) \leq \frac{C}{n \log n}
\] (4.63)
for a constant \( C > 0 \).
Proof. Choose $\varepsilon = \frac{1}{\log n}$ and $\alpha = \frac{8}{n}$, then
\[
\frac{\sqrt{n} - \frac{4}{\varepsilon^2}}{2\varepsilon^2 (\alpha + \pi)} \geq 1
\] (4.64)
for $n \geq 6$ and
\[
\frac{\alpha (n\varepsilon^2 - 8)}{2\varepsilon^2 (\alpha + \pi)} \geq 1
\] (4.65)
for $n \geq 15$. Therefore,
\[
\min_{1 \leq i,j,k \leq n} d(v_i, v_j, v_k) \leq \frac{1}{\log n} \sin \frac{8}{n} \leq \frac{8}{n \log n}
\] (4.66)
for $n \geq 15$, and then (4.63) follows.

We used quasi-exact hard analysis to obtain the decay rate. However, the tools or techniques in hard analysis may be used to obtain the same order of decay but probably better constant in the decay rate. From the perspective of hard analysis, based on numerical experiment results, we formulate the following conjecture for a slower decay rate.

Conjecture 4.12. Let $P_1, \ldots, P_n$ be a set of points in the unit disk on the plane and $d(i, j, k)$ be the distance of the point $P_i$ to the line connecting other two points $P_j$ and $P_k$, $1 \leq i, j, k \leq n$, then
\[
\min_{1 \leq i,j,k \leq n} d(i, j, k) \leq \frac{C}{n^{1+\varepsilon_0}}
\] (4.67)
for some absolute constant $C, C > 0$, independent of $n$ and some $\varepsilon_0 > 0$.

5. Numerical Experiments

In this section, we would like to present some numerical experimental results.

In the first and second numerical experiments, we use MATLAB to randomly generate $n$ points in a unit square $[0, 1]^2$ whose two coordinates are independent and identically distributed copies of uniformly distributed random variables and then compute the minimal distance of a point to the line connecting other two points. For each matrix size $n$, we repeat this procedure $n^2$ times to include $n^2$ sets of points of size $n$, and then take the maximum of the minimal distance over the $n^2$ repeats of randomly generating $n$ points, due to the configurations increase greatly as the size of the point increases. After that, we multiply the maximum of the minimal distance by $n^2$ to compare the decay rate with $\frac{1}{n^2}$. From Figs. 2(a) and 2(b), we can see that $n^2 \min_{1 \leq i,j,k \leq n} d(i, j, k)$ is bounded, as $n$ increases, so $\min_{1 \leq i,j,k \leq n} d(i, j, k)$ decays mostly at the order of at least $O\left(\frac{1}{n^2}\right)$ if the points are generated by normal random variables.

In the third and fourth numerical experiments, we use MATLAB to randomly generate $n$ points in a unit square $[0, 1]^2$ whose two coordinates are independent
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Fig. 2. Plotted above are the smallest distances to lines multiplied by the square of the sizes of point sets, in which the two coordinates of points are independent and identically distributed copies of uniformly distributed random variables.

Fig. 3. Plotted above are the smallest distances to lines multiplied by the square of the sizes of point sets, in which the two coordinates of points are independent and identically distributed copies uniformly distributed random variables.

and identically distributed copies uniformly distributed random variables and then compute the minimal the distance of a point to the line connecting other two points. For each matrix size \( n \), we repeat this procedure \( n^2 \) times to include \( n^2 \) sets of points of size \( n \), and then take the maximum of the minimal distance over the \( n^2 \) repeats of randomly generating \( n \) points, due to the configurations increase greatly as the size of the point increases. After that, we multiply the maximum of the minimal distance by \( n^3 \) to compare the decay rate with \( \frac{1}{n^3} \). From Figs. 3(a) and 3(b), we can see that

\[
\min_{1 \leq i, j, k \leq n} d(i, j, k)
\]

is bounded, as \( n \) increases, so \( \min_{1 \leq i, j, k \leq n} d(i, j, k) \) decays with high probability at the order of \( O\left(\frac{1}{n^3}\right) \) mostly if the points are generated by normal random variables.

In the fifth numerical experiment, we use MATLAB to randomly generate \( n \) points in a unit square \([0, 1]^2\) whose two coordinates are independent and identically distributed copies uniformly distributed random variables and then compute the minimal the distance of a point to the line connecting other two points. For
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Fig. 4. Plotted above are the smallest distances to lines multiplied by the square of the sizes of point sets, in which the two coordinates of points are independent and identically distributed copies uniformly distributed random variables.

From Fig. 4(a), we can see that \( n^3 \min_{1 \leq i, j, k \leq n} d(i, j, k) \) is bounded, as \( n \) increases, so \( \min_{1 \leq i, j, k \leq n} d(i, j, k) \) decays with high probability at the order of \( O(\frac{1}{n}) \) mostly if the points are generated by normal random variables. In the sixth numerical experiment, we use MATLAB to randomly generate \( n \) points in a unit square \([0,1]^2\) whose two coordinates are independent and identically distributed copies uniformly distributed random variables and then compute the minimal the distance of a point to the line connecting other two points. For each matrix size \( n \), we repeat this procedure 100 times to include 100 sets of points of size \( n \), and then take the maximum of the minimal distance over the \( n^2 \) repeats of randomly generating \( n \) points, due to the configurations increase greatly as the size of the point increases.

After that, we multiply the maximum of the minimal distance by \( n^3 \) to compare the decay rate with \( \frac{1}{n^3} \). From Fig. 4(b), we can see that \( n^3 \min_{1 \leq i, j, k \leq n} d(i, j, k) \) is bounded, as \( n \) increases, so \( \min_{1 \leq i, j, k \leq n} d(i, j, k) \) decays with high probability at the order of \( O(\frac{1}{n^3}) \) mostly if the points are generated by normal random variables.

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References